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EDITED BY

ORMOND STONE W. E. BYERLY H. S. WHITE
F. S. WOODS MAXIME BÔCHER
E. V. HUNTINGTON

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THE MATHEMATICAL THEORY OF THE TOP

By A. G. Greenhill

Interest of mathematicians in the theory of the spinning top was stimulated by Professor Klein's *Princeton Lectures* in 1896, where he brought forward the application of his functions a, β, γ, δ, defined in terms of Euler's angles θ, ϕ, ψ by

$$a = \cos\tfrac{1}{2}\theta\, e^{\frac{1}{2}(\psi+\phi)i}, \qquad \delta = \cos\tfrac{1}{2}\theta\, e^{\frac{1}{2}(-\psi-\phi)i},$$
$$\beta = i\sin\tfrac{1}{2}\theta\, e^{\frac{1}{2}(\psi-\phi)i}, \qquad \gamma = i\sin\tfrac{1}{2}\theta\, e^{\frac{1}{2}(-\psi+\phi)i},$$

which give the complete solution in terms of multiplicative elliptic functions of the motion of any point of the symmetrical top; and the theory is receiving its complete development in his work: *Die Theorie des Kreisels*, written in collaboration with Dr. A. Sommerfeld.

In a note, communicated by Dr. Carl Barus to *Science*, Dec. 20, 1901, I submitted a method of arriving at the motion of the axis of the spinning top, applying Poinsot's principle of Angular Momentum and the Hodograph.

It is proposed to resume this treatment in the present article, and to carry on the analysis to the construction of a number of cases in which the results can be expressed in a finite algebraical form, and thus to explore the general analytical field along the easiest lines of progress, which we take to be those on which the parameter which makes its appearance of the Elliptic Integral of the Third Kind is a simple aliquot part of a period, and thereby Abel's theory can be utilized of the pseudo-elliptic integral.*

Some cases of this kind have been investigated already in articles in the *Proceedings of the London Mathematical Society*,† vols. 26 and 27, and illustrated by stereoscopic diagrams drawn by the late Mr. T. I. Dewar; but the degree given there of many of the results may be halved, so the investigation is resumed, and the analysis simplified, the notation also being selected carefully and illustrative diagrams added, drawn accurately to scale to represent an actual case of motion.

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* Abel, *Œuvres*, vol. 1, p. 104, vol. 2, p. 87.

† Abbreviated in the sequel to L. M. S.

1. The top is supposed to be symmetrical about its axis, and spinning with its point in a small, smooth cup O, like the Maxwell top;* as his apparatus is no longer procurable, a bicycle wheel, as in figs. 1 and 3, will be found effective for experimental demonstration.†

The physical constants of the top are given in C. G. S. units by

(i) the weight W in grammes (g), as weighed in a balance;

(ii) the distance h in centimetres (cm) between the point O and the centre of gravity, and then Wh (g-cm) may be called the *preponderance*;

(iii) C_1 and A_1, the moment of inertia (g-cm²) about the axis of figure OC' and about any axis through O at right angles to OC'.

The moment of inertia A_1 can be measured experimentally by swinging the top, without revolution about OC', as a plane or conical pendulum, and observing the length l (cm) of the equivalent simple pendulum, or the angular velocity n (radians/second), or period $2\pi/n$ (seconds), when swung without rotation as a conical pendulum of small angular aperture; then

$$(1)\qquad l = \frac{A_1}{Wh}, \qquad A_1 = Whl,$$

$$(2)\qquad n^2 = \frac{g}{l} = \frac{Wgh}{A_1}.$$

2. Now spin the top about the axis OC'' with angular velocity R (radians/second)—this can be done by means of a stick inserted between the spokes of the bicycle wheel—and denote the angular momentum C_1R by G' and represent it by the vector OC''; this component OC'' of angular momentum G' about the axis OC'' will not be altered by any motion of the axis.

This statement requires closer examination; so hold the axis OC'' of the bicycle wheel in fig. 3 in any position, and then move it round in any closed path back again to its original position, previously marking one of the spokes.

If the wheel is spun with angular velocity R,

$$(1)\qquad \frac{d\phi}{dt} + \cos\theta\,\frac{d\psi}{dt} = R,$$

so that

$$(2)\qquad \phi + \psi = Rt + \int \operatorname{ver}\theta\, d\psi,$$

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* *Scientific Papers*, vol. 1, p. 246.

† C. T. Knipp, *The Physical Review*, vol. 12, 1901.

and in a closed path the last integral is the area a cut out on the unit sphere of centre O by the axis OC'.

If the axis is held fixed again, the wheel in the absence of frictional resistance will continue spinning with angular velocity R, but the marked spoke will have acquired an angular lead a of its position if OC' had been held immovable from the first.

This is shown experimentally with great clearness when the wheel is not spun, so that $R = 0$, as in the Spherical Pendulum case.

3. If the vector OH represents the axis of resultant angular momentum H, when the point O is placed in the cup and the axis of the top is projected in any manner, OH will have components expressed in Euler's angles θ and ψ by (figs. 1 and 2)

$$(1) \qquad OC' = G', \quad C'K = A_1 \sin\theta \frac{d\psi}{dt}, \quad KH = A_1 \frac{d\theta}{dt}.$$

In accordance with the Principle of Angular Momentum the velocity of H is in the line KH parallel to the axis of the torque of gravity, and therefore perpendicular to the vertical plane $OC'K$, so that H will move horizontally and describe a curve in a fixed horizontal plane at a constant height OC above O, the vertical vector OC representing the constant component G of angular momentum about the vertical.

The velocity of H is also equal to the torque of gravity,

$$Wgh \sin\theta = A_1 n^2 \sin\theta \text{ (dyne-cm)};$$

that is, the hodograph of H, turned backwards through a right angle, is similar to the curve described by P, the projection on a horizontal plane of a point on the axis of the top, such as C', and the hodograph vector is

$$A_1 n^2 \sin\theta\, e^{\psi i}.$$

Denoting the polar coordinates of H in the horizontal plane CHK with respect to the origin C by ρ and ω, so that the vector CH is given by $\rho e^{\omega i}$, its vector velocity, turned backwards through a right angle, is

$$(2) \qquad A_1 n^2 \sin\theta\, e^{\psi i} = -i \frac{d}{dt}(\rho e^{\omega i}),$$

by which the projection P of C' is derived from the curve of H by means of a simple differentiation; and this holds also for the general unsymmetrical top, provided we can determine the curve described by H.

Steady Motion

4. In Steady Motion, $d\theta/dt = 0$, H and K coincide, and $CK = \rho$; denoting the constant precessional velocity $d\psi/dt$ by μ, then the gravity torque being equal to the speed of H,

$$(1)\qquad A_1n^2 \sin\theta = \rho\mu = (OC' \sin\theta - C'K \cos\theta)\mu$$
$$= (C_1R \sin\theta - A_1\mu \sin\theta \cos\theta)\mu;$$

and dropping the factor $\sin\theta$,

$$(2)\qquad A_1\mu^2 \cos\theta - C_1R\mu + A_1n^2 = 0,$$

$$(3)\qquad \frac{C_1R}{A_1n} = \frac{\mu}{n}\cos\theta + \frac{n}{\mu} = 2\sqrt{\cos\theta} + \left(\sqrt{\frac{\mu}{n}\cos\theta} - \sqrt{\frac{n}{\mu}}\right)^2,$$

a quadratic for μ when R and θ are assigned for a given top; and the reality of the roots requires the condition

$$(4)\qquad C_1R > 2A_1n\sqrt{\cos\theta}.$$

Interpreted geometrically in fig. 1,

$$(5)\qquad \mu = \frac{A_1n^2 \sin\theta}{CK} = \frac{C'K}{A_1\sin\theta},$$

$$(6)\qquad KC \cdot KC' = A_1^2n^2 \sin^2\theta,$$

$$(7)\qquad KM \cdot KN = A_1^2n^2;$$

or if $C'K$ cuts the vertical OC in L, and the mid-point of $C'L$ is denoted by E,

$$(8)\qquad C'E^2 - EK^2 = C'K \cdot KL = A_1^2n^2 \sin\theta \tan\theta.$$

There is a second position of K, at K_1, and a corresponding second value μ_1 of μ, the larger root of the quadratic (2) for μ; and the critical case occurs when $EK = 0$, and

$$(9)\qquad C'E^2 = \tfrac{1}{4}C'L^2 = \tfrac{1}{4}OC'^2 \tan^2\theta = A_1^2n^2 \sin\theta \tan\theta,$$

$$(10)\qquad C_1^2R^2 = 4A_1^2n^2 \cos\theta, \qquad \frac{C_1R}{A_1n} = 2\sqrt{\cos\theta},$$

as before in (4), and now the quadratic in μ has equal roots

$$(11)\qquad \mu = n\sqrt{\sec\theta} = \frac{2A_1n^2}{C_1R}.$$

When the top is spun rapidly and C_1R/A_1n is great, K is close to C'; and we can take

(12) $$A_1n^2 \sin\theta = \mu \cdot KC = \mu \cdot OC' \sin\theta = C_1R\mu \sin\theta,$$

(13) $$\mu = \frac{A_1n^2}{C_1R},$$

one half the value of μ in (11); the other value μ_1 being great, K_1 being close to L.

5. To gain some idea of the effect of friction, in causing the axis of the top to rise or fall, suppose the axis of resultant angular momentum to change from OK to Ok in consequence of an impulsive couple Kk in the vertical plane KOC; and now calculate the couple Q which, together with the gravity couple $A_1n^2 \sin\theta$, is required to keep the axis OC' moving at the same inclination θ to the vertical.

Then in fig. 1,

(1) $$\frac{A_1n^2 \sin\theta + Q}{ck} = \frac{c'k}{A_1 \sin\theta},$$

(2) $$ck \cdot kc' = A_1^2n^2 \sin^2\theta + A_1Q \sin\theta = CK \cdot KC' + A_1Q \sin\theta,$$

(3) $$\frac{A_1Q}{\sin\theta} = km \cdot kn - KM \cdot KN;$$

so that Q is positive, and the axis requires to be held down in opposition to its tendency to rise, when Kk is towards the concavity of the hyperbola KK_1 having the asymptotes OC, OC'; and *vice versa*.

The resistance of the air for instance to the motion in azimuth may be considered to act as a couple with axis in the direction LC'; so that it will cause the axis to rise or fall according as OK_1 or OK is the angular momentum vector.

On the other hand the couple of resistance to spinning will have its axis in the direction $C'O$, and will cause the axis to fall in both cases.

Unsteady Motion of the Top

6. Returning to the general case of Unsteady Motion, and resolving the velocity of the vector OH in the radial direction CH,

(1) $$\frac{d\rho}{dt} = Wgh \sin\theta \cos CHK = A_1n^2 \sin\theta \frac{KH}{\rho},$$

(2) $$\rho \frac{d\rho}{dt} = Wgh \sin\theta \cdot KH = A_1^2n^2 \sin\theta \frac{d\theta}{dt},$$

and integrating,

(3) $$\tfrac{1}{2}\rho^2 = A_1^2 n^2(E - \cos\theta),$$

where E is a constant.

This relation is also obtainable from the principle of the Conservation of Energy; the sum of the kinetic and potential energy being constant,

(4) $$\tfrac{1}{2}A_1\frac{d\theta^2}{dt^2} + \tfrac{1}{2}A_1\sin^2\theta\frac{d\psi^2}{dt^2} + \tfrac{1}{2}C_1R^2 + Wgh\cos\theta = K,$$

a constant; and thus

(5) $$\rho'^2 = C'H^2 = A_1^2\frac{d\theta^2}{dt^2} + A_1^2\sin^2\theta\frac{d\psi^2}{dt^2} = 2A_1Wgh(D - \cos\theta),$$

where

(6) $$A_1WghD = A_1K - \tfrac{1}{2}A_1C_1R^2;$$

also

(7) $$\rho^2 - \rho'^2 = CH^2 - C'H^2 = CK^2 - C'K^2 = OC'^2 - OC^2 = G'^2 - G^2,$$

so that

(8) $$\rho^2 = 2A_1Wgh(D - \cos\theta) + G'^2 - G^2 = 2A_1^2n^2(E - \cos\theta),$$

where

(9) $$E = D + \frac{G'^2 - G^2}{2A_1^2n^2},$$

(10) $$E + \frac{G^2}{2A_1^2n^2} = D + \frac{G'^2}{2A_1^2n^2} = F$$

suppose; and then

(11) $$OH^2 = 2A_1^2n^2(F - \cos\theta).$$

The quantities G, G', D, E or F may be called the dynamical constants of the problem, as contrasted with W, h, A_1, C_1, n, the physical constants.

7. To make the equations homogeneous, put

(1) $$OC = \delta, \quad OC' = \delta', \quad 4A_1Wgh = 4A_1^2n^2 = k^2;$$

so that

(2) $$\rho^2 = \tfrac{1}{2}k^2(E - \cos\theta), \quad OH^2 = \tfrac{1}{2}k^2(F - \cos\theta),$$

and

(3) $$E + 2\frac{\delta^2}{k^2} = D + 2\frac{\delta'^2}{k^2} = F.$$

Denoting the perpendiculars CK and $C'K$ from C and C' on the tangent at H by p and p',

(4) $$\delta' - \delta \cos\theta = p \sin\theta, \quad \delta - \delta' \cos\theta = p' \sin\theta;$$

also

(5) $$\cos\theta = \frac{Ek^2 - 2\rho^2}{k^2}, \quad \sin\theta = \frac{\sqrt{[k^4 - (Ek^2 - 2\rho^2)^2]}}{k^2},$$

so that, eliminating θ,

(6) $$p^2 = \frac{[\delta' k^2 - \delta(Ek^2 - 2\rho^2)]^2}{k^4 - (Ek^2 - 2\rho^2)^2}.$$

Otherwise,

(7) $$\cos\theta = E - 2\frac{\rho^2}{k^2} = \frac{\delta\delta' - pp'}{OK^2},$$

with

(8) $$OK^2 = p^2 + \delta^2 = p'^2 + \delta'^2;$$

so that, eliminating p',

(9) $$\left[(p^2 + \delta^2)\left(E - 2\frac{\rho^2}{k^2}\right) - \delta\delta'\right]^2 = p^2 p'^2 = p^2(p^2 + \delta^2 - \delta'^2),$$

and dividing out a factor $p^2 + \delta^2$,

(10) $$(p^2 + \delta^2)\left(E - 2\frac{\rho^2}{k^2}\right)^2 - 2\delta\delta'\left(E - 2\frac{\rho^2}{k^2}\right) - p^2 + \delta'^2 = 0,$$

(11) $$p^2 = \frac{\left[\delta' - \delta\left(E - 2\frac{\rho^2}{k^2}\right)\right]^2}{1 - \left(E - 2\frac{\rho^2}{k^2}\right)^2},$$

as before; or

(12) $$OK^2 = \frac{\delta^2 + \delta'^2 - 2\delta\delta'\left(F - 2\frac{OH^2}{k^2}\right)}{1 - \left(F - 2\frac{OH^2}{k^2}\right)^2}.$$

This is found to represent the characteristic property of a Poinsot herpolhode, the curve produced by rolling a quadric surface about its centre, which is fixed, on a fixed tangent plane (the *invariable plane*); as for instance in the motion of a body tossed in the air, when the momental ellipsoid at the centre of gravity rolls about the C. G. on a plane at a fixed distance from the C. G., the plane remaining perpendicular to the axis of resultant angular

momentum of the body; this is shown experimentally in Dr. Hermann Grassmann's *Modelle zur Kreiseltheorie*, constructed by Martin Schilling, Halle a. S.

In the terminology of Routh's *Rigid Dynamics*, vol. 2, §175, OC is the invariable line, and OC' the conjugate line.

The same equation (6) or (12) defines also the trace of a rolling line of curvature, the intersection of a concentric ellipsoid and confocal hyperboloid of two sheets, and then θ will be the variable angle between the generating lines through H of the deformable confocal hyperboloid of one sheet.*

The same curve is also the projection of a tortuous Elastica, as a consequence of Kirchhoff's *Kinetic Analogue*; and the spherical curve of C' is a hodograph of the Elastica, described with constant velocity.

8. Then

(1) $$KH^2 = \rho^2 - p^2 = \frac{R}{k^4 - (Ek^2 - 2\rho^2)^2},$$

where

(2) $$R = \rho^2[k^4 - (Ek^2 - 2\rho^2)^2] - [\delta' k^2 - \delta(Ek^2 - 2\rho^2)]^2,$$
$$= 4(\rho_1^2 - \rho^2)(\rho_2^2 - \rho^2)(\rho_3^2 - \rho^2)$$

suppose, with

(3) $$\rho_1^2 < 0 < \rho_2^2 < \rho^2 < \rho_3^2;$$

and now

(4) $$\frac{d\rho^2}{dt} = 2A_1 n^2 \sin\theta \cdot KH = \frac{n}{k}\sqrt{R},$$

(5) $$nt = \int_{\rho_2}^{\rho} \frac{k\,d\rho^2}{\sqrt{R}},$$

an elliptic integral of the first kind.

Also, denoting the angle KCH by χ,

(6) $$\tan\chi = \frac{\sqrt{(\rho^2 - p^2)}}{p} = \frac{\sqrt{R}}{\delta' k^2 - \delta(Ek^2 - 2\rho^2)},$$

(7) $$\sin\theta\, e^{\chi i} = \frac{\delta' k^2 - \delta(Ek^2 - 2\rho^2) + i\sqrt{R}}{k^2\rho}.$$

Since

(8) $$KH^2 = CH^2 - CK^2,$$

(9) $$A_1^2 \frac{d\theta^2}{dt^2} = 2A_1^2 n^2(E - \cos\theta) - \frac{(G' - G\cos\theta)^2}{\sin^2\theta};$$

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*Cf. Darboux in Despeyrous' *Cours de mécanique*, vol. 2, p 522.

and putting $\cos\theta = z$,

(10) $$\frac{dz^2}{dt^2} = 2n^2 Z,$$

where

(11) $$Z = (E-z)(1-z^2) - \tfrac{1}{2}\left(\frac{G' - Gz}{A_1 n}\right)^2$$
$$= (E-z)(1-z^2) - 2\left(\frac{\delta' - \delta z}{k}\right)^2.$$

Denoting the roots of $Z = 0$ by z_1, z_2, z_3,

(12) $$z_1 + z_2 + z_3 = F,$$

and arranged so that

(13) $$z_1 > 1 > z_2 > z > z_3 > -1,$$

then with $d\theta/dt$ positive and dz/dt negative, as in fig. 1,

(14) $$nt = \int_z^{z_2} \frac{dz}{\sqrt{(2Z)}},$$

an elliptic integral of the first kind, reducible to Legendre's standard form by the substitution

(15) $$z = z_2 \sin^2\varphi + z_3 \cos^2\varphi,$$

so that

(16) $$z_1 - z = (z_1 - z_3)\Delta^2\varphi = (z_1 - z_3)(1 - \kappa^2 \sin^2\varphi),$$
$$z_2 - z = (z_2 - z_3)\cos^2\varphi, \qquad z - z_3 = (z_2 - z_3)\sin^2\varphi$$
$$\kappa^2 = \frac{z_2 - z_3}{z_1 - z_3}, \quad \kappa'^2 = \frac{z_1 - z_2}{z_1 - z_3},$$

and

(17) $$nt = \sqrt{\frac{2}{z_1 - z_3}}\int_\varphi^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = \sqrt{\frac{2}{z_1 - z_3}}\left(K - F\varphi\right),$$

(18) $$F\varphi = K - mt, \text{ where } \frac{m}{n} = \sqrt{\frac{z_1 - z_3}{2}}.$$

Then in Jacobi's notation

(19) $$\varphi = \operatorname{am}(K - mt),$$

and, in Gudermann's notation,

(20) $$z = z_2 \operatorname{sn}^2(K - mt) + z_3 \operatorname{cn}^2(K - mt),$$

and so also

(21) $$\rho^2 = \rho_2^2 \operatorname{sn}^2(K - mt) + \rho_3^2 \operatorname{cn}^2(K - mt).$$

Interpreted dynamically, equation (20) shows that the axis of the top keeps time with the beats of a simple pendulum of length

$$(22)\qquad L = \frac{l}{\frac{1}{2}(z_1 - z_3)},$$

suspended from a point at a height $\frac{1}{2}(z_1 + z_3)l$ vertically above O, in such a manner that a point on the pendulum at a distance

$$(23)\qquad x = \tfrac{1}{2}(z_1 - z_3)l = \frac{l^2}{L}.$$

from its point of suspension keeps at the same level as the point on the axis of the top at a distance l from O, the point of support of the top.

9. Next resolving transversely to CH, or taking the moment of the velocity of H round C,

$$(1)\qquad \rho^2 \frac{d\omega}{dt} = A_1 n^2 \sin\theta \cdot CK = A_1 n^2 (G' - G\cos\theta)$$

$$= \frac{G\rho^2}{2A_1} - A_1 n^2 (GE - G'),$$

so that

$$(2)\qquad \frac{d\omega}{dt} = \frac{G}{2A_1} - \frac{A_1 n^2 (GE - G')}{\rho^2},$$

$$(3)\qquad \omega = \frac{Gt}{2A_1} - A_1 n (GE - G') \int_0^t \frac{n\,dt}{\rho^2} = \frac{\delta}{k} nt - \tfrac{1}{2}\int_{\rho_3}^{\rho} \frac{(\delta E - \delta')k^2 d\rho^2}{\rho^2 \sqrt{R}},$$

involving an elliptic integral of the third kind.

The reduction to the standard form is effected by putting

$$(4)\qquad s - s_a = \tfrac{1}{2}M^2(z - z_a) = M^2 \frac{\rho_a^2 - \rho^2}{k^2}, \qquad a = 1, 2, 3,$$

where M is a homogeneity factor at our disposal; and now from (5) or (14) in §8,

$$(5)\qquad nt = \int_s^{s_2} \frac{M\,ds}{\sqrt{S}}, \qquad S = 4(s - s_1)(s - s_2)(s - s_3);$$

and Weierstrass's elliptic argument u is given by

$$(6)\qquad u = \int_s^{\infty} \frac{ds}{\sqrt{S}} = \int_{s_2}^{\infty} + \int_s^{s_2} = \omega_2 + \frac{nt}{M},$$

so that s is an elliptic function of u which we can denote by $s(u)$, differing from Weierstrass's function $\wp u$ by a constant, while $\sqrt{S}$ is $-\wp' u$.

If v, σ or $s(v)$, Σ or $S(v)$ denote the values of u, s or $s(u)$, S or $S(u)$ corresponding to $z = E$, $\rho^2 = 0$,

$$(7)\qquad \wp v - \wp u = \sigma - s = \tfrac{1}{2}M^2(E - z) = M^2\frac{\rho^2}{k^2},$$

$$(8)\qquad \wp'^2 u = S = \tfrac{1}{2}M^6 Z = M^6\frac{R}{k^6};$$

$$(9)\qquad -i\wp' v = \sqrt{-\Sigma} = M^3\frac{\delta E - \delta'}{k};$$

and the order of arrangement is

$$(10)\qquad z_1 > E > z_2 > z > z_3,$$

$$(11)\qquad \rho_1^2 < 0 < \rho_2^2 < \rho^2 < \rho_3^2;$$

$$(12)\qquad s_1 > \sigma > s_2 > s > s_3;$$

so that, f denoting a real fraction,

$$(13)\qquad v = \omega_1 + f\omega_3, \text{ or } K + fK'i,$$

as it may be written when the argument u or v is supposed to be normalized into Legendre's form.

We may denote $\sqrt{-\Sigma}$ by $2Q(v)$ and now (3) becomes

$$(14)\qquad \omega = \frac{\delta}{k}nt - \int_s^{s_2}\frac{Q(v)ds}{(\sigma - s)\sqrt{S}}.$$

10. We choose as the standard form of the corresponding elliptic integral of the third kind,

$$(1)\qquad I(v) = \int_s^{s_2}\frac{-P(v)(\sigma - s) + Q(v)}{\sigma - s}\frac{ds}{\sqrt{S}},$$

where $P(v)$ is a function chosen so as to make

$$(2)\qquad I(v) = \tfrac{1}{2}i\log\frac{\theta(u + v)}{\theta(u - v)},$$

when u and v are normalized to Legendre's form; and thus

$$(3)\qquad iP(v) = \frac{\eta_3}{\omega_3}v - \zeta v,$$

and in terms of Jacobi's Zeta function,

(4) $$\frac{P(v)}{\sqrt{(s_1 - s_3)}} = \mathrm{Z}(fK', \kappa') = \mathrm{zn} fK',$$

in Glaisher's notation.

Now

(5) $$\omega = \frac{\delta}{k} nt - P(v) \int_s^{s_2} \frac{ds}{\sqrt{S}} - I(v),$$

(6) $$I(v) = pt - \omega,$$

where

(7) $$\frac{p}{n} = \frac{\delta}{k} - \frac{P(v)}{M}.$$

It is convenient to write I, P, Q in the sequel for $I(v)$, $P(v)$, $Q(v)$; also to put

(8) $$x = \sigma - s = \tfrac{1}{2}M^2(E - z) = M^2\frac{\rho^2}{k^2},$$

and, following Darboux, to put

(9) $$\frac{\delta}{k} = \frac{L}{M}, \quad \frac{\delta'}{k} = \frac{L'}{M},$$

(10) $$Z = (E - z)(1 - z^2) - 2\left(\frac{L' - Lz}{M}\right)^2;$$

so that

(11) $$nt = \int_{x_2}^{x} \frac{M dx}{\sqrt{X}}, \quad pt = (L - P)\int_{x_2}^{x} \frac{dx}{\sqrt{X}},$$

(12) $$X = 4(x_3 - x)(x - x_2)(x - x_1),$$
$$= 4(-x^3 + Ax^2 - Bx - Q^2) = S = \tfrac{1}{2}M^6 Z,$$

(13) $$x_1 < 0 < x_2 < x < x_3;$$

(14) $$x_3 - x = (x_3 - x_2)\mathrm{sn}^2(K - mt),$$
$$x - x_2 = (x_3 - x_2)\mathrm{cn}^2(K - mt),$$
$$x - x_1 = (x_3 - x_1)\mathrm{dn}^2(K - mt),$$
$$mt = \sqrt{x_3 - x_1}\,\frac{nt}{M},$$

(15) $$I = \int_{x_2}^{x} \frac{-Px + Q}{x} \frac{dx}{\sqrt{X}}.$$

Now, with

(16) $$z_1 + z_2 + z_3 = F = E + 2\frac{L^2}{M^2} = D + 2\frac{L'^2}{M^2},$$

(17) $$3\wp v = x_1 + x_2 + x_3 = A = \tfrac{1}{2}M^2(3E - F) = M^2E - L^2,$$

(18) $$M^2E = L^2 + A,$$

(19) $$M^2F = 3(L^2 + \wp v),$$

(20) $$M^2z = M^2E - 2x = L^2 + A - 2x,$$

(21) $$-i\wp' v = \sqrt{(-4x_1x_2x_3)} = 2Q = M^3\left(E\frac{\delta}{k} - \frac{\delta'}{k}\right)$$
$$= M^3\left(E\frac{L}{M} - \frac{L'}{M}\right) = M^2EL - M^2L',$$

(22) $$M^2L' = M^2EL - 2Q = L^3 + AL - 2Q = L^3 + 3L\wp v + i\wp' v,$$

(23) $$2\wp'' v = 4(x_2x_3 + x_3x_1 + x_1x_2) = 4B = M^4(3E^2 - 2EF + z_2z_3 + z_3z_1 + z_1z_2)$$
$$= M^4\left(E^2 - 4E\frac{L^2}{M^2} + 4\frac{L'}{M^2} - 1\right)$$
$$= M^4(E^2 - 1) - 4LM^2(LE - L') = M^4(E^2 - 1) - 8LQ,$$

so that

(24) $$M^4 = (L^2 + A)^2 - 8LQ - 4B,$$

thus determining the homogeneity factor M when the data are L and $\sigma = s(v)$; and now

(25) $$\frac{G}{2A_1n} = \frac{\delta}{k} = \frac{L}{M},$$

(26) $$\frac{G'}{2A_1n} = \frac{\delta'}{k} = \frac{L'}{M} = \frac{L^3 + AL - 2Q}{M^3}.$$

11. The angle χ, defined in (6) §8, is also given by

(1) $$\tan\chi = \frac{KH}{CK} = \frac{A_1\sin\theta\frac{d\theta}{dt}}{CK\sin\theta} = \frac{A_1n\sqrt{2Z}}{G' - Gz},$$

(2) $$\sin\theta\, e^{\chi i} = \frac{2\dfrac{L' - Lz}{M} + i\sqrt{2Z}}{\sqrt{[2(E - z)]}},$$

and

(3) $$\frac{L' - Lz}{M} = 2\,\frac{Lx - Q}{M^3},$$

(4) $$2Z = 4X/M^6,$$

(5) $$2(E - z) = 4x/M^2,$$

so that

(6) $$\tfrac{1}{2}M^2 \sin\theta\, e^{\chi i} = \frac{Lx - Q + i\sqrt{\tfrac{1}{4}X}}{\sqrt{x}},$$

and

(7) $$(\tfrac{1}{2}M^2 \sin\theta)^2 = \frac{(Lx - Q)^2 - x^3 + Ax^2 - Bx - Q^2}{x}$$

$$= -\,2LQ - B + (L^2 + A)x - x^2.$$

Combined with the relation

(8) $$(\tfrac{1}{2}M^2 z)^2 = [\tfrac{1}{2}(L^2 + A) - x]^2,$$

we obtain by addition

(9) $$\tfrac{1}{4}M^4 = \tfrac{1}{4}(L^2 + A)^2 - 2LQ - B$$

as before, in (24) §10; and this can be written

(10) $$M^4 = (L^2 - x_1 + x_2 + x_3)^2 - 4(L\sqrt{-x_1} + \sqrt{x_2 x_3})^2$$

$$= (L^2 + x_1 - x_2 + x_3)^2 + 4(L\sqrt{x_2} - \sqrt{-x_1 x_3})^2$$

$$= (L^2 + x_1 + x_2 - x_3)^2 + 4(L\sqrt{x_3} - \sqrt{-x_1 x_2})^2.$$

12. Putting

(1) $$z_1 = \operatorname{ch}\theta_1, \quad z_2 = \cos\theta_2, \quad z_3 = \cos\theta_3, \quad \theta_2 < \theta < \theta_3;$$

(2) $$M^2 \operatorname{ch}\theta_1 = L^2 - x_1 + x_2 + x_3,$$

so that

(3) $$M^4 \operatorname{sh}^2\theta_1 = (L^2 - x_1 + x_2 + x_3)^2 - M^4 = 4(L\sqrt{-x_1} + \sqrt{x_2 x_3})^2,$$

(4) $$\tfrac{1}{2}M^2 \operatorname{sh}\theta_1 = L\sqrt{-x_1} + \sqrt{x_2 x_3};$$

and similarly

(5) $$\tfrac{1}{2}M^2 \sin\theta_2 = L\sqrt{x_2} - \sqrt{-x_1 x_3},$$

(6) $$\tfrac{1}{2}M^2 \sin\theta_3 = L\sqrt{x_3} - \sqrt{-x_1 x_2},$$

(7) $$M^2 \cos\theta_2 = L^2 + x_1 - x_2 + x_3,$$

(8) $$M^2 \cos\theta_3 = L^2 + x_1 + x_2 - x_3.$$

13. Now, with

(1) $$\frac{\sqrt{x}}{M} = \frac{\rho}{k},$$

(2) $$\chi = \omega - \psi, \quad I = pt - \omega,$$

we obtain

(3) $$\tfrac{1}{2}M^2 \sin\theta\, e^{(pt-\psi)i} = \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{\sqrt{x}}\, e^{iI}$$
$$= \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}\,\frac{\rho}{k}\, e^{(pt-\omega)i},$$

and thus the vector of the horizontal projection P of a point C' on the axis of the top is determined, when once the curve of H is known, either as its hodograph, or more expeditiously by means of equation (3); and expressed analytically, the curve of H is given by

(4) $$\frac{\rho^2}{k^2} = \frac{\wp v - \wp u}{M^2} = \frac{\sigma(u+v)\sigma(u-v)}{M^2 \sigma^2 u\, \sigma^2 v},$$

or normalized in the Jacobian form

(5) $$\frac{\rho^2}{k^2} = \frac{\theta^2(0)\theta(u+v)\theta(u-v)}{\lambda^2 \theta^2 u\, \theta^2 v},$$

while

(6) $$e^{2Ii} = \frac{\theta(u-v)}{\theta(u+v)}, \qquad I = pt - \omega,$$

so that

(7) $$\frac{\rho}{k}\, e^{(pt-\omega)i} = \frac{\theta(0)\theta(u-v)}{\lambda\theta u\, \theta v},$$

and the right hand side is an algebraical function when v is an aliquot part of a period.

14. We arrive at the same result by the method of the hodograph of equation (2) §3; changing the sign of i,

(1) $$A_1 n^2 \sin\theta\, e^{-\psi i} = i\frac{d}{dt}(\rho e^{-\omega i}) = \rho e^{-\omega i}\left(\frac{d\omega}{dt} + \frac{i}{\rho}\frac{d\rho}{dt}\right);$$

which with

(2) $$\frac{\rho^2}{k^2} = \frac{x}{M^2},\quad \frac{i}{\rho}\frac{d\rho}{dt} = \frac{1}{2}\frac{i}{x}\frac{dx}{dt} = \frac{n}{M}\frac{\frac{1}{2}i\sqrt{X}}{x},$$

(3) $$\frac{d\omega}{dt} = p - \frac{dI}{dx}\frac{dx}{dt} = \frac{L-P}{M}n - \frac{-Px+Q}{x}\frac{n}{M} = \frac{Lx-Q}{x}\frac{n}{M},$$

leads to

(4) $$\tfrac{1}{2}M^2 \sin\theta\, e^{-\psi i} = \frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}\frac{\rho}{k}e^{-\omega i},$$

as before in (3) § 13; and thus the vector of H is turned into the vector of P, the projection of C', by the factor

(5) $$\frac{Lx - Q + \frac{1}{2}i\sqrt{X}}{x}.$$

15. Before proceeding to the discussion of the algebraical cases as the chief object of this paper, we can examine some special results of the general problem.

When the axis passes periodically through the highest or lowest vertical position, the *rosette* curves are produced, described by Professor Klein in the *Bulletin of the American Mathematical Society*, vol. 3.

In such cases

(1) $$L' = \pm L,\quad M^4L'^2 = M^4L^2,\quad D = E,$$

(2) $$(L^3 + AL - 2Q)^2 = L^2(L^2 + A)^2 - 8L^3Q - 4L^2B,$$

leading on reduction to a cubic in L,

(3) $$QL^3 + BL^2 - AQL + Q^2 = 0,$$

which can be factorized into

(4) $$\left(L + \sqrt{\frac{x_2x_3}{-x_1}}\right)\left(L - \sqrt{\frac{-x_1x_3}{x_2}}\right)\left(L - \sqrt{\frac{-x_1x_2}{x_3}}\right) = 0,$$

or

(5) $$(L - Q/x_1)(L - Q/x_2)(L - Q/x_3) = 0.$$

With

$$(6)\qquad L' = L = -\sqrt{\frac{x_2 x_3}{-x_1}}, \qquad Q = Lx_1,$$

$$(7)\qquad \mathrm{sh}\theta_1 = 0, \qquad z_1 = 1,$$

$$(8)\qquad \tfrac{1}{2}M^2(1-z) = x - x_1, \quad M\sin\tfrac{1}{2}\theta = \sqrt{(x-x_1)},$$

$$(9)\qquad M^2 = L^2 - x_1 + x_2 + x_3 = (x_3 - x_1)(x_2 - x_1)/(-x_1);$$

and from (6) §11

$$(10)\qquad M\cos\tfrac{1}{2}\theta\, e^{\chi i} = \frac{L\sqrt{(x-x_1)} + i\sqrt{(x_3-x)(x-x_2)}}{\sqrt{x}},$$

$$(11)\qquad M\cos\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{L\sqrt{(x-x_1)} + i\sqrt{(x_3-x)(x-x_2)}}{\sqrt{x}}\, e^{iI(v)} = e^{iI(v-\omega_1)},$$

$$(12)\qquad \cot\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{e^{iI(v-\omega_1)}}{\sqrt{(x-x_1)}},$$

with

$$(13)\qquad \frac{p}{n} = \frac{L - P(v)}{M} = \frac{P(\omega_1 - v)}{M},$$

because

$$(14)\qquad P(v) + P(\omega_1 - v) = -\sqrt{\frac{x_2 x_3}{-x_1}} = \frac{Q}{x_1}.$$

In this case the axis does not reach the highest vertical position, but describes an intermediate path.

But with

$$(15)\qquad L' = L = \sqrt{\frac{-x_1 x_3}{x^2}}, \qquad Q = Lx_2,$$

$$(16)\qquad \sin\theta_2 = 0, \qquad z_2 = 1,$$

$$(17)\qquad \tfrac{1}{2}M^2(1-z) = x - x_2, \qquad M\sin\tfrac{1}{2}\theta = \sqrt{(x-x_2)},$$

$$(18)\qquad M^2 = L^2 + x_1 - x_2 + x_3 = (x_3 - x_2)(x_2 - x_1)/x_2,$$

$$(19)\qquad M\cos\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{L\sqrt{(x-x_2)} + i\sqrt{(x_3-x)(x-x_1)}}{\sqrt{x}}\, e^{iI} = e^{iI(\omega_2 - v)},$$

$$(20)\qquad \cot\tfrac{1}{2}\theta\, e^{(pt-\psi)i} = \frac{e^{iI(\omega_2 - v)}}{\sqrt{(x-x_2)}},$$

$$(21)\qquad \frac{p}{n} = \frac{P(\omega_2 - v)}{M},$$

giving an upper rosette curve, in which the axis passes periodically through the highest vertical position.

With

(22) $$L' = -L = -\sqrt{\frac{-x_1x_2}{x_3}},\quad Q = Lx_3,$$

(23) $$\sin\theta_3 = 0,\qquad z_3 = -1,$$

(24) $$\tfrac{1}{2}M^2(1+z) = x_3 - x,\qquad M\cos\tfrac{1}{2}\theta = \sqrt{(x_3 - x)},$$

(25) $$M^2 = -L^2 - x_1 - x_2 + x_3 = (x_3 - x_1)(x_3 - x_2)/x_3,$$

(26) $$M\sin\tfrac{1}{2}\theta e^{(pt-\psi)i} = \frac{-L\sqrt{(x_3 - x)} + i\sqrt{(x - x_2)(x - x_1)}}{\sqrt{x}}e^{iI} = e^{iI(\omega_3 - v)},$$

(27) $$\tan\tfrac{1}{2}\theta e^{(pt-\psi)i} = \frac{e^{iI(\omega_3 - v)}}{\sqrt{(x_3 - x)}},$$

(28) $$\frac{p}{n} = \frac{P(v - \omega_3)}{M},$$

giving a lower rosette curve, in which the axis passes periodically through the lowest vertical position.

Expressed in a tabular form we shall find for these rosettes:

	INTERMEDIATE	UPPER	LOWER
$\operatorname{ch}\frac{1}{2}\theta_1 =$	1	$\operatorname{nd}(1-f)K'$	$\operatorname{nd}(1-f)K'$
$\cos\frac{1}{2}\theta_2 =$	$\operatorname{dn}fK'$	1	$\operatorname{dn}fK'$
$\cos\frac{1}{2}\theta_3 =$	$\operatorname{cn}fK'$	$\operatorname{sn}fK'$	0
$\frac{1}{2}\theta_3 =$	$\operatorname{am}fK'$	$\frac{1}{2}\pi - \operatorname{am}fK'$	$\frac{1}{2}\pi$

Stereoscopic representations of an upper and lower rosette curve drawn for a parameter

(29) $$v = K + \tfrac{1}{2}K'i,$$

are shown in the diagram drawn by the late Mr. T. I. Dewar, published in *Engineering*, Oct. 10, 1897.

16. Cusps will exist on the circle $z = z_2$, if

(1) $$z_2 = \frac{G}{G'} = \frac{LM^2}{L^3 + AL - 2Q},$$

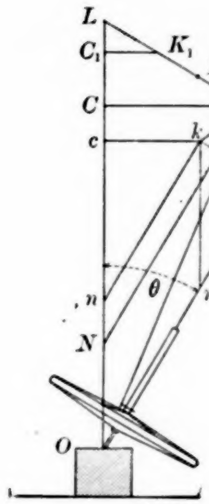

FIG. 1.

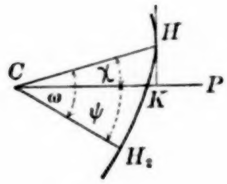

FIG. 2.

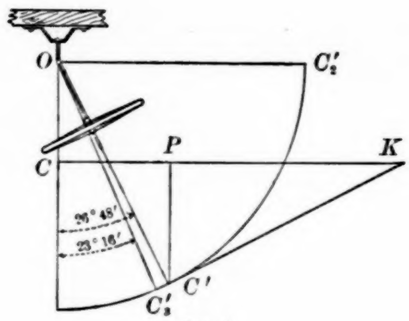

FIG. 3.

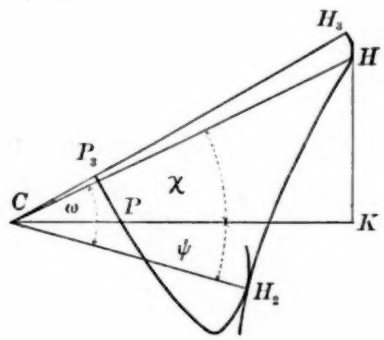

FIG. 4.

$$(2)\qquad (L^2 + x_1 - x_2 + x_3)(L^3 + AL - 2Q) = LM^4$$
$$L = (L^2 + A)^2 - 8L^2Q - 4LB,$$

reducing to

$$(3)\qquad \left(L - \sqrt{\frac{-x_1x_3}{x_2}}\right)\left\{\left(L - \sqrt{\frac{-x_1x_3}{x_2}}\right)^2 - \frac{(x_3 - x_2)(x_2 - x_1)}{x_2}\right\} = 0,$$

of which the second factor must be taken, the first factor giving an upper rosette curve ; and now for cusps

$$(4)\qquad \tfrac{1}{2}M^2 \sin\theta_2 = \sqrt{(x_3 - x_2)(x_2 - x_1)},$$

$$(5)\qquad \tfrac{1}{2}M^2 = \frac{(x_3 - x_2)\sqrt{(-x_1)(x_2 - x_1)} + (x_2 - x_1)\sqrt{x_3(x_3 - x_2)}}{x_2},$$

$$(6)\qquad \sin\theta_2 = \frac{\sqrt{x_3(x_2 - x_1)} - \sqrt{(-x_1)(x_3 - x_2)}}{x_3 - x_1}, = \kappa'(\mathrm{dn}fK' - \kappa\,\mathrm{sn}fK'),$$

$$(7)\qquad \cos\theta_2 = \frac{\sqrt{x_3(x_3 - x_2)} + \sqrt{(-x_1)(x_2 - x_1)}}{x_3 - x_1} = \kappa\,\mathrm{dn}fK' + \kappa'^2\,\mathrm{sn}fK'.$$

When z_1 and z_2 have changed places, cusps can exist on the circle $z = z_1$, and

$$(8)\qquad \left(L + \sqrt{\frac{x_2x_3}{-x_1}}\right)^2 = \frac{(x_3 - x_1)(x_2 - x_1)}{-x_1},$$

with similar expressions for M^2 and θ_1;

$$(9)\qquad \tfrac{1}{2}M^2 = \sqrt{\frac{(x_3 - x_1)(x_2 - x_1)}{-x_1}}\left[\sqrt{x_2(x_3 - x_1)} - \sqrt{x_3(x_2 - x_1)}\right],$$

$$(10)\qquad \sin\theta_1 = \frac{\sqrt{x_2(x_3 - x_1)} + \sqrt{x_3(x_2 - x_1)}}{x_3 - x_2}.$$

To produce cusps experimentally the bicycle wheel after being spun is placed gently in the support O in fig. 1, or the axis in fig. 3 is let go without impulse. If let fall from a height into O a looped figure is produced, or if the axis receives an impulse in a vertical plane.

(To be continued.)

ON QUADRATIC FORMS

By Paul Saurel

In a recent note* we have reproduced the very simple demonstration by means of which Gibbs establishes the conditions for a positive quadratic form. It may be of interest to show that the more general question of determining the number of positive and of negative terms in the expression of a quadratic function as a sum of squares can be treated in a similar manner.

Let us consider the quadratic function φ defined by the equation

$$\varphi = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k, \tag{1}$$

in which

$$a_{ik} = a_{ki}, \tag{2}$$

and let us write

$$f_i = \sum_{k=1}^{n} a_{ik} x_k. \tag{3}$$

From equations 1 and 3 it follows that

$$\varphi = \sum_{i=1}^{n} f_i x_i. \tag{4}$$

If $a_{11} \neq 0$ we can throw this equation into the form

$$\varphi = \frac{f_1^2}{a_{11}} + \sum_{i=2}^{n} f_i^{(1)} x_i, \tag{5}$$

where

$$f_i^{(1)} = \frac{a_{11} f_i - a_{1i} f_1}{a_{11}}. \tag{6}$$

Since $f_i^{(1)}$ is independent of x_1, it follows that the second of the two terms on the right-hand side of equation 5 is independent of x_1. Moreover, it should be noticed that the coefficient of f_1^2 is the reciprocal of

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, x_3, \ldots, x_n}, \tag{7}$$

* Annals of Mathematics, vol. 4 (1902–03), p. 62.

in which the subscripts indicate the quantities which are regarded as independent variables.

If a_{11} be equal to zero, equations 5 and 6 are meaningless. We can, however, obtain a pair of analogous equations by employing any one of the functions f_i for which the corresponding coefficient a_{ii} is not equal to zero. Then by changing the numbering of the variables we can again obtain equations 5 and 6.

We have accordingly the following theorem:

THEOREM I. *If there be one of the quantities*

$$\left(\frac{\partial f_i}{\partial x_i}\right)_{x_1, x_2, x_3, \ldots, x_n}$$

which is not equal to zero, the given quadratic function of n variables can be expressed as the sum of a quadratic function of not more than n—1 *variables and the square of a linear function. The coefficient of this last term has the sign of the non-vanishing derivative.*

If

$$a_{11} = a_{22} = \cdots = a_{nn} = 0, \tag{8}$$

the transformation given above cannot be performed. In this case we shall write

$$\varphi = \frac{2f_1f_2}{a_{12}} + \sum_{i=3}^{n} f_i^{(2)}x_i = \frac{(f_1+f_2)^2 - (f_1-f_2)^2}{2a_{12}} + \sum_{i=3}^{n} f_i^{(2)}x_i, \tag{9}$$

where

$$f_i^{(2)} = \frac{a_{12}f_i - a_{1i}f_2 - a_{2i}f_1}{a_{12}}. \tag{10}$$

Since $f_i^{(2)}$ is independent of x_1 and x_2, it follows that the last term on the right hand side of equation 9 is also independent of x_1 and x_2. If a_{12} be equal to zero the above transformation becomes impossible. In this case, however, an analogous transformation can be performed with a pair of functions f_1, f_i for which the corresponding coefficient a_{1i} is not equal to zero. Such a coefficient exists; otherwise, the given function of n variables would be independent of x_1. The variables can now be renumbered so that equations 9 and 10 are again obtained.

We have accordingly the following theorem:

THEOREM II. *If each of the quantities*

$$\left(\frac{\partial f_i}{\partial x_i}\right)_{x_1, x_2, x_3, \ldots, x_n}$$

be equal to zero, the given quadratic function of n variables can be expressed as the sum of a quadratic function of not more than n—2 variables and the squares of two linear functions. The coefficients of these last two terms have opposite signs.

It is evident that repeated applications of one or both of the transformations just described will yield the following theorem:

THEOREM III. *A quadratic function of n variables can be expressed as the sum of the squares of not more than n linear functions.*

Let us suppose that by repeated applications of Theorems I and II the given quadratic function has been expressed as the sum of r squares and a quadratic function which is independent of $x_1, x_2, \cdots, x_r$. We shall suppose that this quadratic function has been written in each of the forms:

$$\sum_{i=r+1}^{n} f_i^{(r)} x_i, \tag{11}$$

$$\sum_{i=r+1}^{n} \sum_{k=r+1}^{n} a_{ik}^{(r)} x_i x_k . \tag{12}$$

If the last transformation employed be one of the type 5, $f_i^{(r)}$ is defined by the following equation, analogous to equation 6:

$$f_i^{(r)} = \frac{a_{rr}^{(r-1)} f_i^{(r-1)} - a_{ri}^{(r-1)} f_r^{(r-1)}}{a_{rr}^{(r-1)}} . \tag{13}$$

If, on the other hand, the last transformation employed be one of the type 9, $f_i^{(r)}$ is defined by the following equation, analogous to equation 10:

$$f_i^{(r)} = \frac{a_{r-1,r}^{(r-2)} f_i^{(r-2)} - a_{r-1,i}^{(r-2)} f_r^{(r-2)} - a_{r,i}^{(r-2)} f_{r-1}^{(r-2)}}{a_{r-1,r}^{(r-2)}} . \tag{14}$$

In either case

$$a_{ik}^{(r)} = \left(\frac{\partial f_i^{(r)}}{\partial x_k}\right)_{x_{r+1}, x_{r+2}, \ldots, x_n} . \tag{15}$$

We shall now show that

$$a_{ik}^{(r)} = \left(\frac{\partial f_i}{\partial x_k}\right)_{f_1, f_2, \ldots, f_r, x_{r+1}, \ldots, x_n} . \tag{16}$$

Consider the expression

$$\left(\frac{\partial f_i^{(1)}}{\partial x_k}\right)_{x_2,\, x_3,\, \ldots\, ,\, x_n}. \tag{17}$$

Since $f_i^{(1)}$ is independent of x_1 we may give to dx_1 any convenient value. It will therefore be allowable to suppose that dx_1 has been so taken that

$$df_1 = 0. \tag{18}$$

But, in that case, equation 6 shows that

$$df_i^{(1)} = df_i. \tag{19}$$

The expression 17 is therefore equal to

$$\left(\frac{\partial f_i}{\partial x_k}\right)_{f_1,\, x_2,\, x_3,\, \ldots\, ,\, x_n}. \tag{20}$$

Consider next the expression

$$\left(\frac{\partial f_i^{(2)}}{\partial x_k}\right)_{x_3,\, x_4,\, \ldots\, ,\, x_n}, \tag{21}$$

in which $f_i^{(2)}$ is defined by equation 10. Since $f_i^{(2)}$ is independent of x_1 and x_2 we may give to dx_1 and dx_2 any convenient values. It will therefore be allowable to suppose that dx_2 has been so taken that

$$df_1 = 0, \tag{22}$$

and that dx_1 has been so taken that

$$df_2 = 0. \tag{23}$$

But, in that case, equation 10 shows that

$$df_i^{(2)} = df_i. \tag{24}$$

The expression 21 is therefore equal to

$$\left(\frac{\partial f_i}{\partial x_k}\right)_{f_1,\, f_2,\, x_3,\, \ldots\, ,\, x_n}. \tag{25}$$

In a similar manner we can show that

$$\left(\frac{\partial f_i^{(p)}}{\partial x_k}\right)_{f_{p+1}^{(p)},\, f_{p+2}^{(p)},\, \ldots\, ,\, f_r^{(p)},\, x_{r+1},\, \ldots\, ,\, x_n} \qquad i, k > r, \tag{26}$$

is equal to

$$\left(\frac{\partial f_i^{(p-1)}}{\partial x_k}\right)_{f_p^{(p-1)},\, f_{p+1}^{(p-1)},\, f_{p+2}^{(p-1)},\, \ldots,\, f_r^{(p-1)},\, x_{r+1},\, \ldots,\, x_n} \qquad i,\, k > r, \tag{27}$$

or to

$$\left(\frac{\partial f_i^{(p-2)}}{\partial x_k}\right)_{f_{p-1}^{(p-2)},\, f_p^{(p-2)},\, f_{p+1}^{(p-2)},\, f_{p+2}^{(p-2)},\, \ldots,\, f_r^{(p-2)},\, x_{r+1},\, \ldots,\, x_n} \qquad i,\, k > r, \tag{28}$$

according as $f_i^{(p)}$ is defined by an equation of the form 13 or by one of the form 14.

In the first case, since $f_i^{(p)}$ is independent of x_p, we may give to dx_p any convenient value. It will therefore be allowable to suppose that dx_p has been so taken that

$$df_p^{(p-1)} = 0. \tag{29}$$

But then equation 13 shows that

$$df_i^{(p)} = df_i^{(p-1)}, \qquad i > p, \tag{30}$$

and expression 26 transforms at once into expression 27.

In the second case, since $f_i^{(p)}$ is independent of x_{p-1} and x_p, we may give to dx_{p-1} and dx_p any convenient values. It will therefore be allowable to suppose that dx_p has been so taken that

$$df_{p-1}^{(p-2)} = 0, \tag{31}$$

and that dx_{p-1} has been so taken that

$$df_p^{(p-2)} = 0. \tag{32}$$

But then equation 14 shows that

$$df_i^{(p)} = df_i^{(p-2)}, \qquad i > p, \tag{33}$$

and expression 26 transforms at once into expression 28.

It now follows immediately that, by repeated applications of one or both of the equalities just established, equation 15 can be transformed into equation 16.

Equation 16 taken in connection with Theorems I and II enables us to state the following fundamental theorem:

THEOREM IV. *Let us suppose the variables to be so numbered that, by means of the two transformations described above, $x_1, x_2, \ldots, x_r$ can be successively eliminated from the given quadratic function, one or two at a time, and let us further suppose that every one of the derivatives*

$$\left(\frac{\partial f_i}{\partial x_k}\right)_{f_1,\, f_2,\, \ldots,\, f_r,\, x_{r+1},\, \ldots,\, x_n} \qquad i,\, k > r \tag{34}$$

is equal to zero. Then the given quadratic function can be expressed as the sum of r squares. The signs of these r squares are the same as the signs of the following r derivatives:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, \ldots, x_n},$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, \ldots, x_n}, \tag{35}$$

$$\cdots\cdots\cdots$$

$$\left(\frac{\partial f_r}{\partial x_r}\right)_{f_1, f_2, \ldots, f_{r-1}, x_r, \ldots, x_n},$$

provided none of these derivatives be equal to zero. If, however, one of these derivatives be equal to zero, we shall strike it and the following derivative from the list; for every pair of derivatives thus omitted we shall have one positive and one negative square.

The derivatives which appear in the statement of this theorem can be put into another form. Denote by Δ_r the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{vmatrix}.$$

If we make use of equations 3 we obtain without difficulty the following equations:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, \ldots, x_n} = \Delta_1,$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, \ldots, x_n} = \frac{\Delta_2}{\Delta_1}, \tag{36}$$

$$\cdots\cdots\cdots$$

$$\left(\frac{\partial f_r}{\partial x_r}\right)_{f_1, f_2, \ldots, f_{r-1}, x_r, \ldots, x_n} = \frac{\Delta_r}{\Delta_{r-1}}.$$

We obtain also the equation

$$\left(\frac{\partial f_i}{\partial x_k}\right)_{f_1, f_2, \ldots, f_r, x_{r+1}, \ldots, x_n} = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2k} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{rk} \\ a_{i1} & a_{i2} & \cdots & a_{ir} & a_{ik} \end{vmatrix}}{\Delta_r}, \quad i, k > r. \tag{37}$$

If one of the derivatives in 35, the pth for example, be equal to zero, we shall have from equations 36

$$\Delta_p = 0.$$

Moreover, since it was possible to choose as independent variables $f_1, f_2, \ldots, f_{p-1}, x_p, \ldots, x_n$, it follows from equations 3 that

$$\Delta_{p-1} \neq 0.$$

If the derivative on the left side of equation 37 be equal to zero, it follows that the numerator of the right side must be equal to zero. Moreover, since it was possible to choose $f_1, f_2, \cdots, f_r, x_{r+1}, \cdots, x_n$ as independent variables, it follows from equations 3 that

$$\Delta_r \neq 0.$$

Finally, it is well known from the theory of linear equations* that, if the determinant in the numerator of the right side of equation 37 be equal to zero for all values of i and k, then every minor of order $r+1$, and consequently every minor of higher order, that can be formed from the determinant Δ_n will be equal to zero.

These results enable us to restate Theorem IV in the following well-known form:†

THEOREM V. *Consider the series of terms*

$$1, \Delta_1, \Delta_2, \cdots, \Delta_r, \cdots, \Delta_n.$$

It is possible to number the variables so that

$$\Delta_{r+1} = \Delta_{r+2} = \cdots = \Delta_n = 0, \tag{38}$$

* *Cf.* Weber, *Lehrbuch der Algebra*, vol. 1, p. 103.

† *Cf.* Weber, *l. c.*, p. 295.

and so that no two consecutive terms of the series

$$1,\ \Delta_1,\ \Delta_2,\ \cdots,\ \Delta_r \tag{39}$$

shall be equal to zero, the last term itself being different from zero. ***Then the given quadratic function can be expressed as the sum of r squares. The number of positive squares will be the same as the number of permanences in sign in the series 39 and the number of negative squares will be the same as the number of variations in sign in the same series, provided none of the terms in that series be equal to zero. If, however, one of the terms of the series be equal to zero we shall pay no attention to the two adjacent intervals; to every such pair of intervals there will correspond one positive and one negative square.***

The following theorem is well known:*

If a quadratic function be expressed as a sum of squares of independent linear functions, the number of positive terms is always the same and similarly the number of negative terms is always the same, whatever be the mode of transformation employed.

To complete our discussion we must accordingly show that the linear functions obtained by repeated applications of our two transformations are independent. For this purpose it is sufficient to show that one determinant of order r formed from the coefficients of these linear functions does not reduce to zero.† Such a determinant can be formed by taking the coefficients of $x_1, x_2, \cdots, x_r$ in these functions; for it is easy to see that this determinant is equal to the product of non-vanishing factors of the form $a^{(p)}_{p+1,\,p+1}$ and of the form

$$\begin{vmatrix} a^{(p)}_{p+1,\,p+2}, & a^{(p)}_{p+1,\,p+2} \\ a^{(p)}_{p+1,\,p+2}, & -a^{(p)}_{p+1,\,p+2} \end{vmatrix}.$$

New York, March 1903.

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* *Cf.* Weber, *l. c.*, p. 213.

† *Cf.* Weber, *l. c.*, p. 108; Bôcher, Annals of Mathematics, vol. 2 (1900–01), p. 84.

A GENERALIZED CONCEPTION OF AREA: APPLICATIONS TO COLLINEATIONS IN THE PLANE

By Edwin Bidwell Wilson

In a paper presented to the summer meeting, 1902, of the Deutsche Mathematiker-Vereinigung and to be published in the *Jahresbericht* of that society, I pointed out the way in which one might arrive at a conception and numerical measure of volume in a linear space of any number of dimensions without the use of the conception or measure of length. The object of the present article is to discuss from a different point of view this same problem for a space of two dimensions, the plane; and to give some applications of the new notion of area thus obtained to results already known and to some that it is hoped may be new.

The reasons which render justifiable some change in the conception of area become apparent on examining the most evident properties of the successive subgroups of the general projective group. The general group G has eight degrees of freedom. There is a subgroup A generally known as the affine group from the fact that it leaves the line at infinity fixed as a whole though admitting a redistribution of the individual points upon that line. This group has six degrees of freedom. This freedom is cut down to five degrees in passing to the next subgroup L which possesses the characteristic property of conservation of area and which will occupy much attention in the following pages. Should one desire to introduce length it becomes necessary, if the language of non-euclidean geometry be used for a moment, to fix a system of two points, ordinarily the circular points, upon the line at infinity. The subgroup of L which is thus obtained is the group R of rigid motions; provided these be understood to include changes of symmetry in addition to the truly mechanical motions. This group R has three degrees of freedom only. To study the properties of the group L as much as possible by means of its characteristic invariant, *i. e.* area, and with a minimum reference to the more

special invariant, *i. e.* length, of the subgroup R may be a worthy object in itself and may be expected in some small way to lead to valuable points of view as in a much larger way the use of the cross-ratio in treating the general group G is more suggestive than the direct employment of length.

1. Equivalence of area and its generalization. In dealing with area the triangle is taken as the element. If the vertex of a triangle be displaced parallel to the base, the area understood in the ordinary manner is unchanged. By displacing one vertex parallel to its opposite side, then another parallel to the opposite side of the triangle thus obtained, and by continuing this process of displacement, any given triangle ABC may be carried over into a triangle $A_1B_1C_1$ quite different. It is not difficult to grant that by properly carrying out the construction any triangle ABC may be transferred into any other triangle $A'B'C'$, provided only that the areas of the two triangles have the same magnitude and the same sign, that is the same cyclic order of the letters.

The idea therefore which enters necessarily into the conception of equivalence of triangles and which at the same time is independent of the concept of length is the idea of parallel lines and only that. Acting on this clew the following definition for the generalized equivalence of triangular areas will be laid down.

Definition: Two triangles ABC and $A'B'C'$, neither of which cuts the line l or touches it, are said to be equivalent in area *with respect to that line* l when and only when one triangle may be transformed into the other by displacing a vertex along the line joining that vertex to the intersection of the opposite side with the line l and by repeating this operation successively, upon the various intermediary triangles thus formed, a finite number of times. (See figure 1.)

A glance at the figure shows that the successive triangles in the sequence are ABC, ABC', $A'BC'$, $A'B'C'$, and that they preserve the same cyclic order of the letters. The line l in the conception of generalized equivalence plays the same rôle as the line at infinity in the ordinary conception. It is also evident that if $A'B'C'$ is equivalent to ABC, then inversely ABC is equivalent to $A'B'C'$; for the construction can be carried out indifferently backward or forward. For similar reasons the statement that two triangles equivalent to the same triangle are equivalent to each other becomes obvious.

As a matter of definition two triangles which have the same base and whose two opposite vertices are collinear with the point of intersection of the

base and the line l are equivalent. And now, if that definition is to be useful, it is necessary that conversely two triangles ABC and ABC'' which have the same base AB and whose vertices are not collinear with the intersection of AB and l shall be non-equivalent. From another standpoint this is merely the demand that the axiom "the whole is not equivalent to its part" shall hold for our generalization of the conception of area. For it is possible to construct a triangle $AB\overline{C}$ which shall be equivalent to ABC'' and hence to ABC and such that the point $\overline{C}$ lies upon the line AC. If the point $\overline{C}$ does not coincide with C one of the triangles ABC and $AB\overline{C}$ must contain the other and the

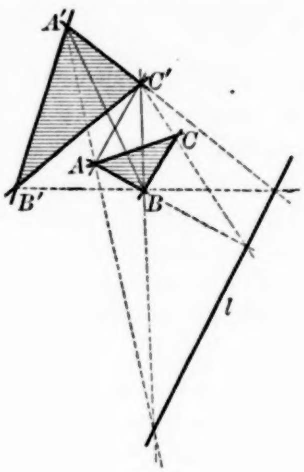

FIG. 1.

whole would be equivalent to its part. It becomes necessary to prove that no such case can arise out of the definition.

To give a geometrical proof apparently offers grave difficulties owing to the large number of steps which may be inserted between any initial triangle and its final equivalent. In the paper referred to above I gave the proof analytically, without in anyway introducing the conception or measure of length, by employing a system of projective coordinates dependent upon the projective definition of a number-system, which in turn is deduced from harmonic constructions. Although the germ of this number-system and system of coordinates may possibly be seen lurking in the geometrical nets of Hamilton and Möbius, yet the first to give them even a partially satisfactory development was von Staudt in his *Beiträge zur Geometrie der Lage*. The treatment has been

amplified and bettered by his followers Klein*, Killing†, Pasch‡, Schur§. These matters are however somewhat recondite. Here we shall follow the plan adopted frequently in the presentation of synthetic projective geometry and practically always in that of analytic projective geometry. We shall have recourse to the ordinary conceptions of area, for convincing ourselves of the truth of some fundamental properties of the generalized area.

For example, if in the case under discussion it were possible to carry the triangle ABC into $AB\overline{C}$ as explained above, then if the line l be projected to infinity, it would be possible by parallel line constructions to carry a triangle $A_1B_1C_1$, the projection of ABC, into $A_1B_1\overline{C}_1$, the projection of $AB\overline{C}$. This is an absurdity inasmuch as the triangles $A_1B_1C_1$ and $A_1B_1\overline{C}_1$ cannot have the same area in the ordinary sense of the word. Thus the difficulty of the part being equivalent to the whole is disposed of even in the generalized notion of area with respect to a line l.

Theorem 1. A triangle ABC is equivalent to each of the triangles CAB, BCA obtained by permuting the letters cyclically.

Draw through B and C respectively lines cutting the line l in its intersections with AB and AC. Let these lines intersect in C'. The following sequence of equivalent triangles demonstrates the equivalence between ABC and CAB:

$$ABC = ABC' = CBC' = CAC' = CAB.$$

The equivalence between ABC and BCA is proved in the same manner.

2. Non-equivalence of generalized area. For the immediate present to simplify matters it will be supposed that the line l to which area is referred is in a very remote region of the plane or even at infinity. This restriction will be considered and removed later where the questions connected more subtly with the ideas of the sign and infinite magnitude of areas will be more easily handled than at present.

Definition. The area of a triangle is said to be reversed in sign when the cyclic order of the letters is changed; *i. e.*, $ABC = -ACB = -CBA = -BAC$. In general two triangles ABC and $A'B'C'$ are said to be equivalent

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* *Math. Annalen*, vol. 37; *Nicht-Euclidische Geometrie* (autographierte Vorlesungshefte), vol. 2.

† *Grundlagen der Geometrie*, vol. 2, pp. 107 et seq., pp. 119 et seq.

‡ *Neuere Geometrie*.

§ *Math. Annalen*, vol. 53.

negatively when one may be carried into the other with the cyclic order of the letters reversed.

THEOREM II. Two triangles equivalent negatively to a third are equivalent (positively) to each other.

THEOREM III. The necessary and sufficient condition that two triangles ABC and ABC'' upon the same base AB shall be equivalent negatively is that the line CC'' shall be divided harmonically by AB and the line of reference l.

First, if the triangles are negatively equivalent displace the vertex $\overline{C}'$ along the line joining C'' to M, the intersection of l and AB, until it falls upon the point $\overline{C}'$ such that $A\overline{C}'$ and CB meet on l. Join $C\overline{C}'$. Then

$$-ABC'' = AC''B = A\overline{C}'B = A\overline{C}'C.$$

By supposition ABC' and ABC are negatively equivalent. Hence $A\overline{C}'C$ is equivalent to ABC. Hence BC'' and AC meet on l. $ABC\overline{C}''$ is therefore a complete quadrangle. The opposite sides $C\overline{C}'$ and AB are divided harmonically by the diagonal l. The pencil composed of AB, l, MC, $M\overline{C}'$ is harmonic and the transversal CC'' is divided harmonically by AB and l. The necessity of the condition announced in the theorem has thus been shown. The sufficiency may be proved in a converse manner.

As a corollary of Theorems II and III it follows that if two triangles ABC, ACD have the same vertex, a common side AC, and bases in the same straight line BCD, the necessary and sufficient condition that they be equal is that BD be divided harmonically by C and the line l.

If moreover upon the base produced of a given triangle ABC we construct the fourth harmonic D of B with respect to C and l, and then the fourth harmonic E of C with respect to D and l, and so on, the series of triangles

$$ABC, \quad ACD, \quad ADE, \; \cdots$$

will all be mutually equivalent. And it may be mentioned that the points $B, C, D, E, \cdots$ to the point M in which BC cuts l, are the points which may be taken to correspond to the integers $0, 1, 2, 3, \ldots, \infty$ in the projective number system above referred to.

It may also be seen that the area ABM must needs be infinite if the area ABC be considered other than zero. As it would be unnatural to denominate any two-dimensional triangle ABC of zero area, the result follows that any triangle MNP which has one vertex P upon the line l is of infinite area.

For, an arbitrary triangle MNQ, where Q lies upon NP, may be taken as of unit area and the triangle MNP can not then be divided into a finite number of triangles equivalent to MNQ. As an extension, any triangle MNP which cuts the line l in any manner cannot be finite in area with respect to that line l. The reason therefore appears, as perhaps it has not yet appeared, why the restricting clause " neither of which cuts the line l or touches it " was inserted in the definition of equivalence.

To divide a triangle ABN which does not cut the line l into n equivalent parts by lines through the vertex A is an operation dependent upon the system of integers just mentioned. Choose at random any line q passing through B and not coincident with BC. Choose arbitrarily a point C_1 corresponding to the number 1. On the line q construct, as above described, the integers up to n, using the point in which q cuts l as the point ∞ of the line q. Let these integers be represented by $C_1, D_1, \ldots N_1$. Let NN_1 cut the line l in the point O. Let $OC_1, OD_1 \ldots$ cut BN in $C, D, \ldots$ respectively. The triangles ABC, $ACD, \ldots$ are then obviously equivalent and each may be regarded as the nth part of ABN.

The areas of two triangles ABC and $A'B'C'$ may be compared to any degree of numerical approximation by a method well known in elementary geometry. The triangle ABC may be divided into n equivalent parts of which one is ABB_1. Upon $A'B'$ a triangle $A'B'B_1'$ may be constructed equivalent to ABB_1. The triangle $A'B'C'$ may then be filled up with k triangles $A'B'B_1'$, $A'B_1'B_2'$, $A'B_2'B_3'$, $\ldots$, $A'B_{k-1}'B_k'$, which are equivalent, and one triangle $A'B_k'C'$, which is smaller. The ratio of ABC to $A'B'C'$ is n to k approximately — the approximation being less than one part in n. If the two triangles are commensurable two numbers n, k may be found such that the ratio is exactly n to k with no remainder. If the triangles are incommensurable the ratio n to k, as n is taken indefinitely greater and greater, will approach an irrational number as a limit.

A definite triangle $U_1U_2U_3$ may be assumed arbitrarily as a triangle of unit area with respect to the line l. By methods of comparison explained above, the numerical value of the area of any triangle ABC relatively to $U_1U_2U_3$ may be determined. As in the ordinary conception of area, the area of a figure not triangular must be determined by dividing that area into a finite or infinite number of triangles.

Thus the first object of this paper, namely, to explain the ideas of area with respect to a line and of the numerical measure of that area has been ac-

complished. There remains only the removal of the restriction imposed at the beginning of this section.

Consider any three lines a, b, c which do not all pass through the same point. These three lines divide the plane into four distinct regions such that it is possible to pass from any point of one region to any other point of that region without crossing any of the three lines, although crossing the line at infinity may be necessary. To pass from one region of the plane to another it is necessary to cross one side of the triangle. A line l which does not pass through any of the three points of intersection of the lines a, b, c cuts across three and only three of these four regions. There remains always one region, and only one, which has no point in common with l and consequently is a triangle not excluded from consideration by the definition of equivalence. If two such triangles are given their relative signs might give difficulty. The fact, too, that one or both of these triangles may be of what we should ordinarily regard as infinite extent is the only other difficulty. This latter has already been spoken of. The former may be removed by considering the sense (positive or negative) of the ranges in which the sides of the triangle cut the line of reference l. For example let the triangles be given by the lines a, b, c and a', b', c'. Let a, b, c cut l in A, B, C respectively; and let a', b', c' cut it in A', B', C'. The triangles will be said to have the same sign or opposite signs relatively to l according as the ranges ABC and $A'B'C'$ upon l have the same or opposite sense. Hence evidently two triangles which lie wholly on the same side of l have the same sign if their vertices follow in the same cyclic order; two triangles which lie wholly on opposite sides of l have the same sign if their vertices follow in opposite cyclic orders. For triangles which lie partly on one side, partly on the other,— that is, for triangles which cross the line at infinity,— no such simple rule can be stated. It is only safe to consult the senses of the ranges which the sides cut out on l.

3. Constructions and miscellaneous applications. There will occur so many occasions in the course of this section to speak of two lines which have the property of meeting on the line l with respect to which area is measured, that some special designation will be convenient. Two such lines will be said to be *parallel with respect to l*. It will generally suffice to abbreviate the expression to simply *parallel*, leaving the reference to the line l to be understood; for the line at infinity and parallelism in the elementary sense of the word will not, one may almost say cannot, be of importance in the subject under discussion here.

Construction. To construct a triangle upon a given base $A'B'$ which shall be equivalent to a given triangle ABC. (See figure 2.)

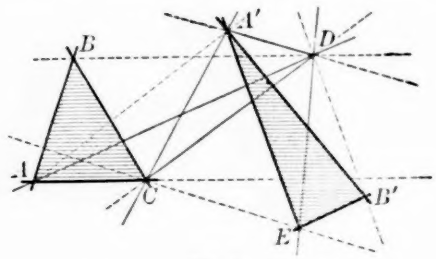


FIG. 2.
In this figure the line l has been taken at infinity for convenience in drawing.

Join AA'. From C draw a line parallel (in the extended sense) to AA'; and from B a line parallel to AC. Let them intersect in D. Join DB' and DA'. From A' draw a line parallel to DB'; and from C a line parallel to $A'D$. Let them intersect in E. Then

$$ABC = ADC = A'DC = A'DE = A'B'E.$$

The number of steps required to pass from one triangle to another is thus seen to be four. The construction fails when D cannot be found except on the line l; that is, when A' lies on the side AC. As A' can not lie on all three of the sides AC, CB, BA, it is always possible to carry the triangle ABC into a triangle upon the base $A'B'$, though that side $A'B'$ may correspond to AC or CB instead of to AB. Of course it must be remembered that A' and B' can not be situated on the line l and further that the segment $A'B'$ is that portion of the whole line $A'B'$ which does not cut l. Figure 3 shows an example of

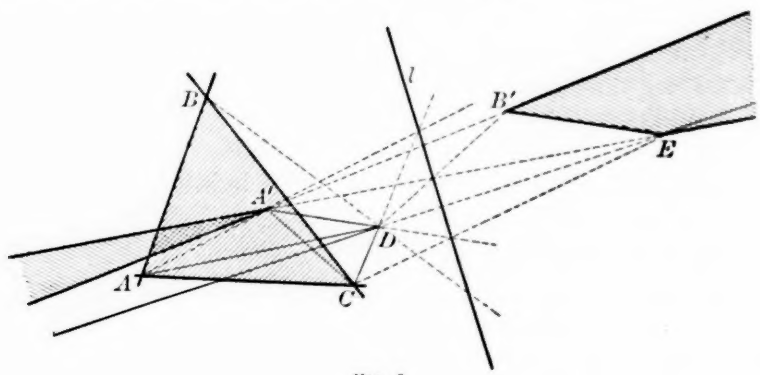


FIG. 3

this case. In the succession of triangles $A'DE$ is not shaded owing to the confusion it would introduce with $A'B'E$. If the construction is performed in the inverse order the triangle $A'B'E$ will be carried over into ABC. Thus there is obtained an example of the construction of a triangle equivalent to a given triangle which lies on two sides of l but does not cut l.

THEOREM IV. The envelope of a line which cuts off a constant area from two given fixed lines is a conic tangent to those two lines at their points of intersection with the line of reference l. (See figure 4.)

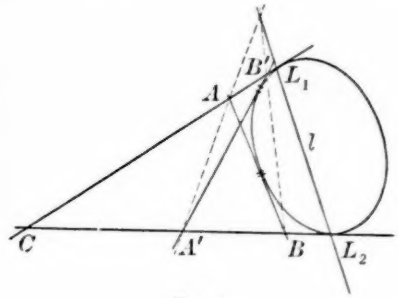


FIG. 4.

Let ACB and $B'CA'$ be two positions of the triangle. Join $A'A$ and BB'. As the areas of the triangles are equal the differences between those areas and the common area ACA' are equal. Therefore

$$AA'B' = AA'B.$$

The vertices B and B' must lie on a line passing through the intersection of the common base $A'A$ and the line l. If then the points of intersection of l with the fixed lines be L_1 and L_2, the ranges

$$CA'BL_2 \text{ and } CAB'L_1$$

are perspective. By interchanging the position of the points in both point-pairs of a given range the resulting range is projective to the given range. Hence

$$CAB'L_2 \overline{\wedge} L_1B'AC.$$

Hence

$$CA'BL_2 \overline{\overline{\wedge}} CAB'L_1 \overline{\wedge} L_1B'AC.$$

The lines joining corresponding points of two projective ranges envelope a conic which is tangent to the lines on which the ranges lie at the points which correspond to the point of intersection of the two ranges. Hence $A'B'$ and AB touch a conic tangent to the lines CL_1 and CL_2 at L_1 and L_2.

THEOREM V. A line h moves so as to cut off a constant area from a given curve which has only two real intersections with any line (the curve is oval). Then h is divided harmonically with respect to the given curve by the point of intersection of h and l and by the point of tangency of h to its envelope. (See figure 5.)

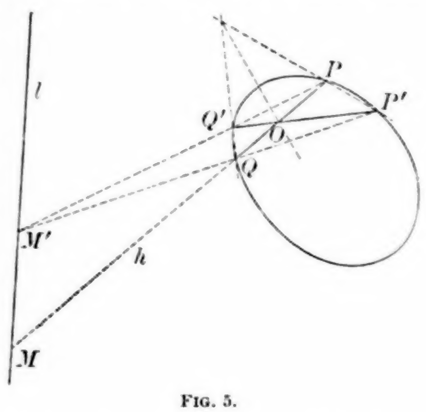


FIG. 5.

Let PQ and $P'Q'$ be two positions of h. Let $P'Q'$ be considered as

approaching PQ as a limit. Let PQ and $P'Q'$ intersect in O. In the limit O is the point of tangency of PQ with its envelope. The sectors PQP', $Q'QP$ are equal. The small curved segments upon the bases PP' and QQ' are insignificant compared with these sectors. Hence the triangles PQP' and $Q'QP$ are practically equal: the limit of their ratio is unity. In the limit therefore the lines PQ' and $P'Q$ meet on l in a point M. By the properties of the complete quadrangle $PP'QQ'$ the points O and M are separated harmonically by the curve. In the figure the point M' has been placed upon the line l to suit the requirements of Theorem VI. In general M' when approaching M will not move along the line l.

This proof of this theorem possesses the same degree of unsatisfactoriness as that of the corresponding theorem in the ordinary treatment of areas. There are numerous points to be filled in, some restrictions to be placed upon the given curve, etc. These difficulties are involved rather in the nature of the subject treated than in the particular method now employed and hence will be waived. By making use of this theorem and those theorems concerning conic sections which are most fundamental from von Staudt's point of view* it is easy to prove the following

Corollary. The envelope of a chord which cuts off a constant area with respect to a line l from a given conic K is a conic K' which is tangent to K in two points, real or imaginary — the chord of contact being the line l. Or, to avoid the possible imaginary case, the statement may be made thus: the polarities which define K and K' set up the same involution upon the line l. In case the line l cuts K the involution is hyperbolic and the points of contact of K and K' are real; in case l does not cut K the involution is elliptic and the points of contact are no longer real: in the intermediate case where l is tangent to K, K and K' have contact of the third order at the point of tangency.

THEOREM VI. The necessary and sufficient condition that a chord PQ cut off a constant area from a conic K is that the range of points P upon the conic shall be projective to the range of points Q upon the conic and that the axis of the projectivity of these two ranges of the second order shall be the line of reference l. (The axis of a projectivity is the line which is the locus of the points of intersection of the lines joining non-corresponding points of any two corresponding pairs of points in the projectivity.)

* See a "Note on the Geometrical Treatment of Conics," by C. A. Scott, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900–01), p. 64.

This theorem is merely another statement of the corollary above. The proof may be given like that of Theorem V. But the most serious difficulties in the theory of limits here vanish of themselves because the point M' of intersection of $P'Q$ and PQ' lies necessarily upon l, the axis of projectivity. Hence the triangles PQP' and $QP'Q'$ are rigorously equal. The demonstration of Theorem VI should therefore be given independently of the more or less unsatisfactorily demonstrated Theorem V.

If two triangles PAB and $PA'B'$ which have the same vertex P and whose bases AB and $A'B'$ lie upon the same line are to be equivalent it is evidently necessary that the segments AB and $A'B'$ shall lie either partly or wholly without one another. Otherwise one triangle would be merely a part of the other. Suppose that the first case obtains. Let the order of the points upon the base line be $AA'BB'$. Then the triangle $PA'B$ is common to the two given triangles and hence PAA' is equivalent to PBB'. The question of the equivalence of two triangles with common vertex and collinear bases, therefore, reduces in every case to an equivalence in which the bases do not overlap.

Theorem VII. The necessary and sufficient condition that two triangles PAB and $PA'B'$, whose bases lie upon the same line but do not overlap, shall be equivalent with respect to a line l not cutting either of them, is that the line l shall pass through the external point of harmonic division of the pairs of points AB' and $A'B$.*

Let O_1 be the internal point of division and O_2 the external point. Then by Theorem III,

$$PBO_1 = -PA'O_1 \text{ and } PAO_1 = -PB'O_1$$

with respect to any line l through O_2. By subtraction

$$PAB = -PB'A' = PA'B'.$$

This proof may be given very neatly without reference to the negative areas upon which Theorem III depends. The construction is as follows. Produce AP until it cuts l in A_1. Draw BA_1 and let it cut PB in C. The triangles PAB and PAC are equivalent. In like manner produce $B'P$ until it cuts l in B_1'. Draw $A'B_1'$ and let it cut PA in C'. Then the triangles $PA'B'$ and

* By the external point will naturally be meant that one through which a line l may be drawn not cutting either of the triangles. Through the other point of harmonic division no line could be drawn not cutting either PAB or $PA'B'$.

$PC'B'$ are equivalent. Let $B'C'$ and AC intersect in H_1; $A'C'$ and BC, in H_2. Owing to the properties of the complete quadrilateral it is evident that PH_1H_2 is a straight line passing through O_1 and that CC' passes through O_2. The ranges

$$AC'PA_1 \overline{\overline{\wedge}} B'CPB'_1 \text{ with respect to } O_2.$$

Hence

$$AC'PA_1 \overline{\wedge} CB'B'_1P.$$

The triangles PAC and $PC'B'$ are therefore equal with respect to l by Theorem IV.

4. Application to the point-line involutory collineation. The sole involutory collineation in the plane is the point-line reflection. By this is meant the transformation in which a point P is replaced by a point P' such that the line PP' passes always through a fixed point O called the center and is divided harmonically by O and a fixed line p called the axis. This transformation is a projective generalization of ordinary reflection in a mirror. For this reason it will often be found convenient to use the words *projective reflection* to stand for the generalized transformation.

THEOREM VIII. *If a triangle ABC is transformed into $A'B'C'$ by a projective reflection the areas of the triangles ABC and $A'B'C'$ are equal with reference to the axis of the reflection.*

Consider first a triangle AOB and the corresponding triangle $A'OB'$. Let OAA' cut the axis l (or p) in A'', and OBB' in B''. Then the ranges

$$AOA'A'' \quad \text{and} \quad BOB'B''$$

are harmonic. But one pair of points in harmonic range may be interchanged without destroying the harmonic property. Hence

$$AOA'A'' \overline{\wedge} BB''B'O.$$

Hence AB, $A'B'$, OB," OA'' envelop a conic and by Theorem IV the triangles AOB and $A'OB'$ are equivalent. Then in similar manner

$$BOC = B'OC' \quad \text{and} \quad COA = C'OA'.$$

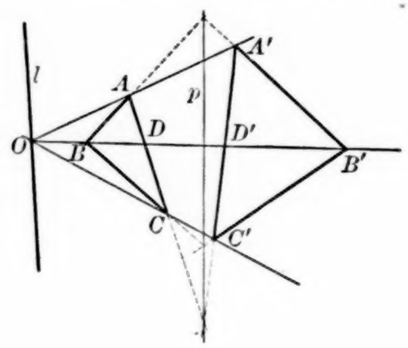
If the sign of the area be taken into consideration

$$-ABC = AOB + BOC + COA = A'OB' + B'OC' + C'OA' = -A'B'C'.$$

The theorem is therefore proved.

THEOREM IX. If a triangle ABC is transformed into $A'B'C'$ by a projective reflection the areas of the triangles ABC and $A'B'C'$ are equal in magnitude and opposite in sign with reference to any line l drawn through the center of the reflection. (See figure 6.)

It is to be understood that the triangle ABC, and consequently the triangle $A'B'C'$, is that one of the four triangles ABC (into which the plane is divided by the lines AB, BC, CA, see p.35) which is not cut by the line l. It will be convenient to assume that the letter B has been assigned to the vertex so situated that the line OB cuts the base AC interiorly at D. Let OB cut $A'C'$ at D'. Inasmuch as AC is transformed into $A'C'$ and the line OB is left fixed by the reflection, it is evident that D and D' are corresponding points and are harmonically separated by O and p. Apply Theorem VI to the triangles into which ABC and $A'B'C'$ are divided by OBB'. By definition of equivalence with respect to l the triangles $D'AB'$ and $D'A'B'$ are equivalent. By Theorem VI the triangles ABD and $AD'B'$ are equivalent since O is one of the pairs of points which separate BB' and DD' harmonically. Hence



$$A'D'B' = AB'B' = ABD = -ADB;$$

and likewise

$$C'D'B' = CD'B' = CBD = -CDB;$$

Subtracting,

$$A'B'C' = -ABC.$$

The theorem is therefore proved.

THEOREM X. The necessary and sufficient condition that a collineation which leaves the line l fixed shall transform any triangle ABC into a triangle $A'B'C'$ equivalent to ABC with regard to that line l is that the collineation be resoluble into the product of two projective reflections of which the poles lie upon the line l.

The sufficiency of the condition is immediately seen by aid of Theorem IX. For by the two reflections the area suffers merely two changes of sign and consequently is finally left unaltered. To discuss the necessity of the condi-

tion, suppose that the triangle ABC is transformed into an equivalent triangle $A'B'C'$. The lines AB, BC, CA cut out three points c, a, b on the line of reference l. The lines $A'B'$, $B'C'$, $C'A'$ cut out three points c', a', b'. Let T be the collineation which carries ABC into $A'B'C'$. Then

$$a[T]a', \qquad a[T]b', \qquad c[T]c'.$$

Let S be the linear transformation of the line l, not of the whole plane, which carries a, b, c into a', b', c'. S is known to be resoluble into two involutions.* Let these involutions be s_1, s_2: then

$$S = s_1 s_2; \qquad a, b, c\, [s_1 s_2]\, a', b', c'.$$

The involutions s_1, s_2, each consists in replacing the points of the line l by the harmonic points with respect to a pair of points p_1, O_1 and p_2, O_2. Draw through the points p_1 and p_2 lines p_1 and p_2 and consider the projective reflections t_1 and t_2 which have p_1, O_1, p_2 and O_2 respectively as axis and center. The product $[t_1 t_2]$ produces the same transformation on l as the product $[s_1 s_2]$:

$$a, b, c\, [t_1 t_2]\, a', b', c'.$$

From the product of T and the inverse of $[t_1 t_2]$ which is $[t_2^{-1} t_1^{-1}]$ and apply the transformation to the line l: then

$$a, b, c\, [T]\, a', b', c'\, [t_2^{-1} t_1^{-1}]\, a, b, c,$$

that is,

$$a, b, c\, [T t_2^{-1} t_1^{-1}]\, a, b, c.$$

Let

$$T\, t_2^{-1} t_1^{-1} = R.$$

The transformation R carries the line l identically into itself. The triangle ABC is therefore carried into a triangle $A''B''C''$ such that the corresponding sides meet on l.

It remains to discuss the variations in R as the directions of the lines p_1 and p_2, which must necessarily pass through the points p_1 and p_2 on l, are varied. There are a number of cases to consider. These will be stated in a series of lemmas.

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* The general linear transformation in one dimension is resoluble in ∞^1 ways into the product of two involutions. See a paper on "Collineations of Space" by R. G. Wood, Annals of Mathematics, ser. 2, vol. 2 (1900-01), p. 164, where further references are given.

Lemma 1. If the two centers O_1 and O_2 are distinct a given point P may be carried over by two reflections t_2 and t_1, whose axes pass through two fixed points p_1 and p_2 on l, into any assigned point P''.

For, by t_2 the point P may be carried into any point P' of the line PO_2. Then by t_1, P' may be carried into any point of the line joining $P'O_1$. This point evidently is any point P'' of the entire plane.

Lemma 2. In case the two centers O_1 and O_2 coincide and the two points p_1 and p_2 on l coincide, the point P may be carried over into any arbitrary point P'' by t_2t_1.

For the transformation upon l is in this case the identical transformation. Hence the common points O_1 and O_2, p_1 and p_2 may be chosen arbitrarily. Choose the points O_1 and O_2 as coincident with the point of intersection of the line PP'' with the line l. Choose also the points p_1 and p_2 at any arbitrary common position. Let the line p_2 be any line. Then P is carried into P'. Let the line p_1 be so chosen that P' is carried into P''. For this purpose p_1 and l must separate P' and P'' harmonically.

The case in which O_1 and O_2 coincide but the points p_1 and p_2 do not coincide need not be considered separately from the case of Lemma 1. For whether the points O or the points p be chosen as centers of the projective reflections is evidently quite immaterial as far as the displacement of the points upon the line l is concerned. Hence:

THEOREM XI. It is always possible to choose two projective reflections t_2 and t_1 which first produce any assigned transformation upon the line l joining their centers and secondly carry any preassigned point P into any other assigned point P''.

Consider next the transformation R. Let T carry the point A into A'. Choose t_2 and t_1 so that the product t_2t_1 carries A' into A. Then since t_2 and t_1 are involutory their inverse transformations t_2^{-1} and t^{-1} are equal respectively to t_2 and t_1, and the product $t_2^{-1}t_1^{-1}$ will carry A' into A. Hence

$$ABC\,[Tt_2^{-1}t_1^{-1}]\,AB''C'' \quad \text{or} \quad ABC\,[R]\,AB''C''.$$

It must be remembered that every transformation R leaves the line l identically fixed. Hence AB and AB'', AC and AC'' are collinear and BC and $B''C''$ intersect on l. But by supposition the areas ABC and $A'B'C'$ are equal. As the transformations t_2 and t_1 do not alter area with regard to their line of centers l, the areas ABC and $AB''C''$ are equal. Hence if $B''C''$ and

BC lie between the point A and the line l they are necessarily coincident and if they are separated by A and l they are necessarily divided harmonically by A and l.

Lemma 3. If the rôle of the points p_1 and O_1 upon the line l be interchanged—that is if the axis p_1 of the reflection be taken to pass through O_1 and the center of the reflection be taken at p_1 and if this transformation be designated by u to distinguish it from t_2, then if t_2t_1 carry a triangle ABC into $\bar{A}\bar{B}\bar{C}$ a transformation t_2u may be found which carries ABC into $\bar{A}B'C'$ in such a manner that $\bar{B}$ and B', $\bar{C}$ and C' are harmonically separated by $\bar{A}'$ and l.

In the first place, by Lemmas 1 and 2 it is always possible to find a transformation t_2u which shall carry A into $\bar{A}$. Secondly, since t_2t_1 and t_2u produce the same transformation on l, the lines $\bar{A}\bar{B}B'$ and $\bar{A}\bar{C}C'$ are straight. Furthermore the lines $\bar{B}\bar{C}$ and $B'C'$ meet on l. The areas of the triangles $\bar{A}\bar{B}\bar{C}$ and $\bar{A}B'C'$ are each equal to that of ABC and hence are equal to each other. Therefore either $\bar{B}\bar{C}$ and $B'C'$ coincide or they are divided harmonically by the point $\bar{A}$ and the line l. Suppose they coincide. Then the transformations t_2t_1 and t_2u carry the triangle ABC into the triangle $\bar{A}\bar{B}\bar{C}$ and leave the line l fixed. The transformations are therefore equal. It must be remembered that t_1 and t_2 and u have been used for *any* projective reflections which have their axes passing through the points p_1, p_2, O_1 and their centers situated at O_1, O_2, p_1 respectively. The letter t_2 thus stands for a type of transformations rather than for any particular individual. Hence the t_2 in t_2t_1 need not be equal to the t_2 in t_2u. To designate this difference affix an accent to the second t_2. Then, on the supposition made above,

$$t_2t_1 = t'_2u.$$

Multiplying by t'_2 and remembering that the square of an involutory transformation is unity,

$$t'_2t_2t_1 = t'_2t'_2u = u.$$

Multiplying by t_1,

$$t'_2t_2 = ut_1.$$

The transformation ut_1 has the characteristic property that the axis of each of its factors passes through the center of the other. Let the axes intersect in some point Q. Then the transformation ut_1 is a projective reflection of which the center is Q and the axis l. For let P be any point of the plane. Let P be carried by u into P' and let P' be carried by t_1 into P''. Then the line QP is carried into QP''. But as the line l is identically fixed QPP'' is

straight. Moreover by the harmonic property of the transformation t_1 it is evident that P and P'' are separated harmonically by Q and l. Therefore the transformation ut_1 is a projective reflection with Q as center and l as axis. Denote ut_1 by v; then

$$t_2' t_2 = ut_1 = v.$$

This equation states that the product of two transformations t_2 which have the same center O_2 and whose axes pass through the same point p_1 on l is an involutory transformation v — which is impossible, as a moment's consideration will render evident. Hence the supposition that t_2t_1 and t_2u carried ABC into the same triangle $\overline{ABC}$ is impossible and the contrary supposition, which is the statement of the lemma, stands proved.

To turn back again to the consideration of the transformation R. It was seen that R carried the triangle ABC of Theorem X into a triangle $AB'C'$ which was either identical with ABC or the harmonic counterpart of it with regard to A and l. If this latter case obtains it is only necessary to replace t_2t_1 by $t_2'u$ and the resulting transformation

$$R' = Tt_2'^{-1}u^{-1}$$

carries ABC into ABC identically. In the former case R itself carries ABC into ABC identically. The transformations R and R' leave l fixed. Hence R or R', as the case may be, is the identical transformation. Let t be a transformation which may stand indifferently for t_2 or t_2' and let u stand likewise for t_1 or u as the case may require. Then the result of the investigations since the statement of Theorem X is

$$Tt^{-1}u^{-1} = 1,$$

or

$$T = ut.$$

Hence Theorem X is proved. With the statement of this theorem in a little different form this paper will close: *The necessary and sufficient condition that a collineation of the plane shall be resoluble into two projective reflections is that the collineation shall leave areas unaltered when referred to one of the fixed lines of the collineation.*

École Normale Supérieure
Paris, France,
December, 1902.

LINES OF CURVATURE ON MINIMUM DEVELOPABLES

By Frederick S. Woods

In the treatises on Differential Geometry are usually found one or both of the two following theorems, different in form but the same in content:

1. If every line upon a surface is a line of curvature, the surface is a plane or a sphere; and conversely.

2. If for any surface the fundamental quantities of the first order are in a constant ratio to those of the second order, the surface is a plane or a sphere; and conversely.

These theorems are true for real surfaces; but in general no restriction to real surfaces is made either in stating or proving the theorems, although the proofs are generally given with the use of explicit or suppressed hypotheses which rule out of consideration one class of imaginary surfaces, namely the developable surfaces which contain the imaginary circle at infinity. Accordingly it seems to have escaped notice that these surfaces, which we shall call minimum developables, should be named with the plane and the sphere in the enunciation of the above theorems.

We define a line of curvature on a surface as any curve such that the normals to the surface taken along the curve form a developable surface. Since we do not restrict ourselves to real quantities, the geometric terms employed have analytic definitions. A surface is defined by the equation

$$z = f(x, y),$$

where $f(x, y)$, together with its derivatives of at least three orders, is finite and continuous within the (x, y)-region considered. We write, as usual,

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

The tangent plane at any point is defined by the equation

$$p(X - x) + q(Y - y) - (Z - z) = 0,$$

and the normal line by the equations

$$\frac{X - x}{p} = \frac{Y - y}{q} = \frac{Z - z}{-1}.$$

The necessary and sufficient condition that the surface should be a minimum developable is

$$p^2 + q^2 + 1 = 0,$$

and the above equations show that the normal line then lies in the tangent plane and is in fact a generator of the minimum developable. It is clear then that any line on the minimum developable satisfies the definition of a line of curvature, for the normals to the surface form a developable, namely the minimum developable itself.

The differential equation of the lines of curvature is

$$[pqr - s(1 + p^2)]dx^2 + [r(1 + q^2) - t(1 + p^2)]dxdy \\ + [pqt - s(1 + q^2)]dy^2 = 0,$$

and consequently all lines upon a surface are lines of curvature when and only when

$$\begin{aligned} pqr &- s(1 + p^2) = 0, \\ r(1 + q^2) &- t(1 + p^2) = 0, \\ pqt &- s(1 + q^2) = 0. \end{aligned} \tag{1}$$

An easy calculation shows that these conditions are satisfied if the surface is a plane, a sphere, or a minimum developable. Conversely we shall show that the above relations lead necessarily to one of these three surfaces.

(1) If $r = s = t = 0$, equations (1) are satisfied, and integration gives a plane

$$z = ax + by + c.$$

Further, if any one of the three quantities, r, s, t, is zero, the others are also, as may readily be seen.

(2) Assuming $r \pm 0$, $s \pm 0$, $t \pm 0$, we may write the equations (1) in the form

$$\frac{pq}{s} = \frac{1 + p^2}{r} = \frac{1 + q^2}{t} = \phi,$$

or

$$pq = s\phi, \tag{2}$$
$$1 + p^2 = r\phi, \tag{3}$$
$$1 + q^2 = t\phi, \tag{4}$$

where ϕ is an unknown function of x and y.

These equations being satisfied by all points on the surface, we have necessarily

$$rq + ps = \phi \frac{\partial s}{\partial x} + s\frac{\partial \phi}{\partial x}, \tag{5}$$

$$2ps = \phi \frac{\partial r}{\partial y} + r\frac{\partial \phi}{\partial y}, \tag{6}$$

which are obtained by differentiating (2) with respect to x and (3) with respect to y. Remembering that $\frac{\partial s}{\partial x} = \frac{\partial r}{\partial y}$, we obtain from (5) and (6),

$$rq - ps = s\frac{\partial \phi}{\partial x} - r\frac{\partial \phi}{\partial y}\cdot \tag{7}$$

Similarly, by differentiating (2) with respect to y, (4) with respect to x, and subtracting one result from the other, we find

$$-sq + pt = -t\frac{\partial \phi}{\partial x} + s\frac{\partial \phi}{\partial y}\cdot \tag{8}$$

Hence, from (7) and (8),

$$p(rt - s^2) = -(rt - s^2)\frac{\partial \phi}{\partial x},$$

$$q(rt - s^2) = -(rt - s^2)\frac{\partial \phi}{\partial y},$$

and therefore either

$$(2a) \qquad rt - s^2 = 0,$$

or

$$(2b) \qquad \frac{\partial \phi}{\partial x} = -p, \quad \frac{\partial \phi}{\partial y} = -q.$$

(2a) The equation $rt - s^2 = 0$, is the condition that a surface should be a developable surface. From (1) we have also

$$1 + p^2 + q^2 = 0.$$

The developable is therefore a minimum developable.

(2b) The equations

$$\frac{\partial \phi}{\partial x} = -p, \quad \frac{\partial \phi}{\partial y} = -q$$

give at once

$$\phi = c - z.$$

This value substituted in (2), (3), and (4) gives relations which are readily transformed into

$$\frac{\partial}{\partial x}(pz - cp) = \frac{\partial}{\partial y}(qz - cq) = -1,$$

$$\frac{\partial}{\partial y}(pz - cp) = \frac{\partial}{\partial x}(qz - cq) = 0;$$

whence

$$pz - cp = a - x,$$
$$qz - cq = b - y;$$

and therefore

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = d^2,$$

the equation of a sphere.

We see then that equations (1) lead necessarily to a plane, a sphere, or a minimum developable, and we have already seen that each of these surfaces furnishes a true solution of the equations.

In considering the second of the theorems stated at the beginning of this article, we need to notice a difference of usage in regard to the fundamental quantities of a surface. If the equations of the surface are

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

there is essential agreement in using as the fundamental quantities of the first order

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Gauss introduced as fundamental quantities of the second order

$$D = A\frac{\partial^2 x}{\partial u^2} + B\frac{\partial^2 y}{\partial u^2} + C\frac{\partial^2 z}{\partial u^2},$$

$$D' = A\frac{\partial^2 x}{\partial u \partial v} + B\frac{\partial^2 y}{\partial u \partial v} + C\frac{\partial^2 z}{\partial u \partial v},$$

$$D'' = A\frac{\partial^2 x}{\partial v^2} + B\frac{\partial^2 y}{\partial v^2} + C\frac{\partial^2 z}{\partial v^2},$$

where

$$A = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \quad B = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \quad C = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}.$$

Subsequent writers have used these quantities, divided by $\sqrt{EG - F^2}$.

Now the necessary and sufficient condition that the surface should be a minimum developable is that $EG - F^2 = 0$, and hence for such a surface the fundamental quantities of the second order as last defined are meaningless. We shall accordingly use Gauss's definitions. The problem is to find surfaces for which

$$\frac{E}{D} = \frac{F}{D'} = \frac{G}{D''}.$$

If we place $u = x$, $v = y$, as we may without loss of generality, these equations become equations (1) already discussed.

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